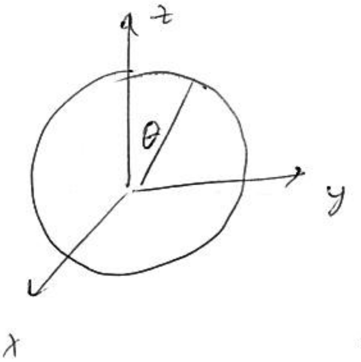
Решавање Лапласове и Поасонове јн

Временски независни потенцијали $\varphi, \vec{A}$; $\Delta\varphi = -\frac{1}{\varepsilon_0}\rho$; $\Delta\vec{A} = -\mu_0\vec{J}$

1. По површини сфере радијуса R распоређено је наелектрисање по закону $\sigma = \sigma_0\cos\theta$, где је θ поларни угао сферног координатног система чији је координатни почетак у центру сфере. Нађи потенцијал и јачину електричног поља у свакој тачки простора.

$\mathrm{div}\vec{E} = \frac{\rho}{\varepsilon_0}$ $\quad \mathrm{div}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$ Л. гејџ $\quad \mathrm{div}\left(-\mathrm{grad}\varphi - \frac{\partial\vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon_0}$

$-\Delta\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{\rho}{\varepsilon_0} \Rightarrow \Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$



$\Delta V = 0$ Нема запреминских наелектрисања.

$V = V(r,\theta) = F(r)\cos\theta \rightarrow$ претпоставка

$$\Delta V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}$$

$$\Delta V = \Delta\left(F(r)\cos\theta\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right)\cos\theta + \frac{F(r)}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta(-\sin\theta)\right)$$

$$= \frac{1}{r^2}\,2r\frac{dF}{dr}\cdot\cos\theta + \frac{d^2F}{dr^2}\cos\theta + \frac{F(r)}{r^2\cancel{\sin\theta}}(-2\cancel{\sin\theta})\cos\theta$$

$$= \cos\theta\left(\frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} - \frac{2}{r^2}F(r)\right) = 0$$

$$\Rightarrow \boxed{F'' + \frac{2}{r}F' - \frac{2}{r^2}F = 0}$$ Ојлерова диференцијална јна

$F \sim r^\alpha \qquad \alpha(\alpha-1)\cancel{r^{\alpha-2}} + 2\alpha\cancel{r^{\alpha-2}} - 2\cancel{r^{\alpha-2}} = 0$

$\alpha^2 + \alpha - 2 = 0 \Rightarrow \alpha = 1$ или $\alpha = -2$

$r<R \qquad F_<(r) = Ar + \frac{B}{r^2} \qquad\qquad r>R \qquad F_>(r) = Cr + \frac{D}{r^2}$

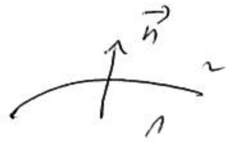
Дивергирање пот. у ∞ и 0 $\Rightarrow$ $B=0$ и $C=0$

$$V = \begin{cases} Ar\cos\theta, & r<R \\ \frac{D}{r^2}\cos\theta, & r\geq R \end{cases}$$

Одређивање константи из граничних услова:

1) непрекидност потенцијала

2) $E_{2n} - E_{1n}\big|_{гр.} = \frac{\sigma}{\varepsilon_0}$ $\qquad E_{2\tau} = E_{1\tau}\big|_{гр}$



1) $V_<\big|_{r=R} = V_>\big|_{r=R}$ $\qquad AR\cancel{\cos\theta} = \frac{D}{R^2}\cancel{\cos\theta} \Rightarrow \boxed{D = AR^3}$

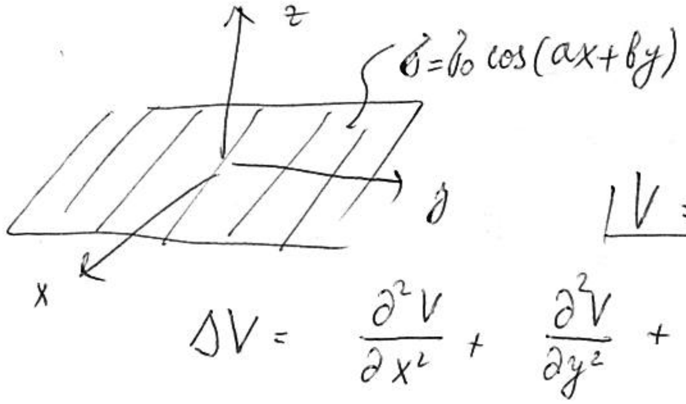
2) $$E_{<n} = -\frac{\partial V_<}{\partial r} = -A\cos\theta$$
$$E_{>n} = -\frac{\partial V_>}{\partial r} = \frac{2D}{r^3}\cos\theta$$
$$\boxed{\frac{2D}{R^3}\cancel{\cos\theta} + A\cancel{\cos\theta} = \frac{\sigma_0 \cancel{\cos\theta}}{\varepsilon_0}}$$

$$\frac{3D}{R^3} = \frac{\sigma_0}{\varepsilon_0} \Rightarrow \boxed{D = \frac{\sigma_0 R^3}{3\varepsilon_0} \; ; \; A = \frac{\sigma_0}{3\varepsilon_0}}$$

$$\boxed{V = \begin{cases} \frac{\sigma_0}{3\varepsilon_0} r\cos\theta, & r \le R \\ \frac{\sigma_0}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta, & r > R \end{cases}} \qquad \vec E = -\mathrm{grad}\, V = -\frac{\partial V}{\partial r}\vec e_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\vec e_\theta$$

$$\boxed{\vec E = \begin{cases} -\frac{\sigma}{3\varepsilon_0}(\cos\theta\,\vec e_r - \sin\theta\,\vec e_\theta), & r \le R \\ -\frac{\sigma_0}{3\varepsilon_0}\frac{R^3}{r^3}(-2\cos\theta\,\vec e_r - \sin\theta\,\vec e_\theta), & r > R \end{cases}}$$

2. xy раван наелектрисана је густином наелектрисања $\sigma = \sigma_0\cos(ax+by)$, где су a и b константе. Наћи потенцијал електричног поља у свакој тачки простора као и јачину електричног поља



Решавамо јну $\Delta V = 0$

$\boxed{V = F(z)\cos(ax+by)}$ претпостављено решење

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$
$$= -F(z)(a^2+b^2)\cos(ax+by) + \frac{d^2F}{dz^2}\cos(ax+by) = 0$$

$$\boxed{F'' - (a^2+b^2)F = 0} \qquad F \sim e^{\pm\sqrt{a^2+b^2}\,z}$$

$z<0 \quad V_< \quad ; \quad z\geq 0 \quad V_>$

$$V_< = A e^{\sqrt{a^2+b^2}\,z}\cos(ax+by) \qquad V_> = B e^{-\sqrt{a^2+b^2}\,z}\cos(ax+by)$$

$$V_< = V_> \Big|_{z=0} \Rightarrow A = B$$

$$E_{>n} - E_{<n} \Big|_{z=0} = \frac{\sigma}{\varepsilon_0}$$

$$-\frac{\partial V_>}{\partial z} + \frac{\partial V_<}{\partial z}\Big|_{z=0} = \frac{\sigma}{\varepsilon_0}$$

$$\sqrt{a^2+b^2}\,A\cos(ax+by) + \sqrt{a^2+b^2}\,A\cos(ax+by) = \frac{\sigma_0}{\varepsilon_0}\cos(ax+by)$$

$$A = \frac{\sigma_0}{2\varepsilon_0\sqrt{a^2+b^2}}$$

$$V = \begin{cases} \frac{\sigma}{2\varepsilon_0\sqrt{a^2+b^2}} e^{\sqrt{a^2+b^2}\,z}\cos(ax+by), & z\leq 0 \\ \frac{\sigma}{2\varepsilon_0\sqrt{a^2+b^2}} e^{-\sqrt{a^2+b^2}\,z}\cos(ax+by), & z\geq 0 \end{cases} = \boxed{\frac{\sigma}{2\varepsilon_0\sqrt{a^2+b^2}} e^{-\sqrt{a^2+b^2}|z|}\cos(ax+by)}$$

$$\boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0\sqrt{a^2+b^2}} e^{-\sqrt{a^2+b^2}|z|}\left(+\sin(ax+by)(a\vec{e}_x + b\vec{e}_y) + \sqrt{a^2+b^2}\,\mathrm{sgn}z\cos(ax+by)\,\vec{e}_z\right)}$$

8. Запреминска густина наелектрисања у цилиндричним координатама има облик $\rho = \begin{cases} \rho_0\left(\frac{r}{R}\right)^n\cos(n\varphi), & r\leq R \\ \rho_0\left(\frac{R}{r}\right)^n\cos(n\varphi), & r>R \end{cases}$, где су ρ_0 и R позитивне константе, а $n\in\mathbb{N}$. Наћи потенцијал у свакој тачки простора. $\boxed{n>1}$

$$\boxed{\Delta V = -\frac{1}{\varepsilon_0}\rho}$$ Поасонова јна

$$V = F(r)\cos(n\varphi) \qquad \Delta V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\frac{1}{r}\frac{d}{dr}(rF')\cancel{\cos(n\varphi)} + \frac{1}{r^2}(-n^2)F(r)\cancel{\cos(n\varphi)} = -\frac{1}{\varepsilon_0}\rho_0\left(\frac{r}{R}\right)^{n\cdot\mathrm{sgn}(R-r)}\cancel{\cos(n\varphi)}$$

$$F'' + \frac{1}{r}F' - \frac{n^2}{r^2}F = -\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{R}\right)^{+n\,\mathrm{sgn}(R-r)}$$ Ојлерова диф. јна

$$F = F_h + F_p \qquad F_h \sim r^\alpha \qquad \alpha(\alpha-1)+\alpha-n^2 = 0 \Rightarrow \alpha = \pm n$$

$$F_p = C\,r^{n\,\mathrm{sgn}(R-r)+2}$$

$$C\cdot(n\,\mathrm{sgn}(R-r)+2)(n\,\mathrm{sgn}(R-r)+1) + C(n\,\mathrm{sgn}(R-r)+2) - Cn^2 = -\frac{\rho_0}{\varepsilon_0}R^{-n\,\mathrm{sgn}(R-r)}$$

$$C\,4(n\,\mathrm{sgn}(R-r)+1) = -\frac{\rho_0}{\varepsilon_0}R^{-n\,\mathrm{sgn}(R-r)}$$

$$C = -\frac{\rho_0}{4\varepsilon_0(n\,\mathrm{sgn}(R-r)+1)}R^{-n\,\mathrm{sgn}(R-r)} \qquad F_p = \begin{cases} -\frac{\rho_0}{4\varepsilon_0(n+1)}R^{-n}r^{n+2}, & r<R \\ -\frac{\rho_0}{4\varepsilon_0(-n+1)}R^{n}r^{-n+2}, & r>R \end{cases}$$

$$V = \cos n\varphi \begin{cases} A r^{n} - \dfrac{\rho_0}{4\varepsilon_0(+n+1)} R^{-n} r^{n+2}, & r<R \\ B r^{-n} - \dfrac{\rho_0}{4\varepsilon_0(-n+1)} R^{n} r^{-n+2}, & r\geq R \end{cases}$$

$$1^\circ \quad V_< = V_> \Big|_{r=R} : \quad A R^{n} - \frac{\rho_0}{4\varepsilon_0(n+1)} R^2 = B R^{-n} - \frac{\rho_0}{4\varepsilon_0(-n+1)} R^2$$

$$\Rightarrow \quad B = A R^{2n} + \frac{\rho_0 R^{2+n}}{4\varepsilon_0(n^2-1)}(-2n).$$

$$2^\circ \quad E_< = E_> \Big|_{r=R} \quad n A R^{n-1} - \frac{\rho_0}{4\varepsilon_0(n+1)}(n+2)R = -n\frac{B}{R^{n+1}} - \frac{\rho_0}{4\varepsilon_0(-n+1)}(-n+2)R$$

$$\cancel{n} A R^{n-1} - \frac{\rho_0 R(2\cancel{n})}{4\varepsilon_0(n^2-1)} = -\cancel{n} A R^{n-1} + 2n^{\cancel{2}}\frac{\rho_0 R}{4\varepsilon_0(n^2-1)}$$

$$2AR^{n-1} = \frac{\rho_0 R(1+n)}{2\varepsilon_0(n^2-1)} \rightarrow A = \frac{\rho_0 R^{2-n}}{4\varepsilon_0(n-1)} \ ; \quad B = \frac{\rho_0 R^{2+n}}{4\varepsilon_0(n-1)} + \frac{\rho_0 R^{n+2}(-2n)}{4\varepsilon_0(n^2-1)}$$

$$B = -\frac{\rho_0 R^{n+2}}{4\varepsilon_0(n+1)}$$

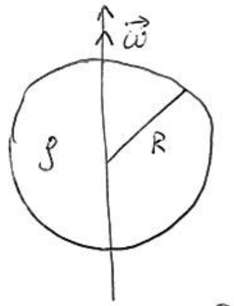
$$\boxed{V = \begin{cases} \left[\dfrac{\rho_0 R^2}{4\varepsilon_0(n-1)}\left(\dfrac{r}{R}\right)^n - \dfrac{\rho_0 r^2}{4\varepsilon_0(n+1)}\left(\dfrac{r}{R}\right)^n\right]\cos n\varphi, & r\leq R \\ \left[-\dfrac{\rho_0 R^2}{4\varepsilon_0(n+1)}\left(\dfrac{R}{r}\right)^n + \dfrac{\rho_0 r^2}{4\varepsilon_0(n-1)}\left(\dfrac{R}{r}\right)^n\right]\cos n\varphi, & r>R \end{cases}}$$

4. Кугла радијуса R, равномерно наелектрисана запреминском густином наелектрисања ρ, ротира око своје осе константном угаоном брзином ω. Наћи векторски потенцијал и јачину магнетног поља у свакој тачки простора.

$$\vec{E} = -\mathrm{grad}\,\varphi - \frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = \mathrm{rot}\,\vec{A}; \quad \mathrm{rot}\,\vec{B} = \mathrm{rot\,rot}\,\vec{A} = \mathrm{grad\,div}\,\vec{A} - \Delta\vec{A} = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\mathrm{grad\,div}\,\vec{A} - \Delta\vec{A} = \mu_0\vec{j} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathrm{grad}\,\varphi - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Rightarrow \Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mathrm{grad}\Big(\mathrm{div}\,\vec{A} + \frac{1}{c^2}\underbrace{\frac{\partial\varphi}{\partial t}}_{=0}\Big) - \mu_0\vec{j} = \mathrm{grad}\big(\overbrace{\partial_\mu A^\mu}^{=0 \text{ Лор. кал.}}\big) - \mu_0\vec{j}$$



$$\Delta\vec{A} = -\mu_0\vec{j}$$

$$\vec{j} = \rho\vec{v} = \rho\,\vec{\omega}\times\vec{r} = \rho\,\omega\vec{e}_z \times (\underbrace{r_c}_{r\sin\theta}\vec{e}_{r_c} + \underbrace{z}_{r\cos\theta}\vec{e}_z) = \rho\omega r_c\vec{e}_\varphi = \rho\omega r\sin\theta\,\vec{e}_\varphi$$

$$\boxed{\vec{A} = f(r)\sin\theta\,\vec{e}_\varphi} \qquad \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y$$

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

$$\Delta\vec{A} = \Delta\big(f(r)\sin\theta(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y)\big) = -\vec{e}_x\,\Delta\big(f(r)\sin\theta\sin\varphi\big) + \vec{e}_y\,\Delta\big(f(r)\sin\theta\cos\varphi\big)$$

$$= -\vec{e}_x\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)\sin\theta\underbrace{\sin\varphi} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\cos\theta)f(r)\underbrace{\sin\varphi} - \frac{1}{r^2\sin^{\cancel{2}}\theta}f(r)\cancel{\sin\theta}\underbrace{\sin\varphi}\right]$$

$$+\vec{e}_y\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)\sin\theta\underbrace{\cos\varphi} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\cos\theta)f(r)\underbrace{\cos\varphi} - \frac{1}{r^2\sin^{\cancel{2}}\theta}f(r)\cancel{\sin\theta}\underbrace{\cos\varphi}\right]$$

$$= \vec{e}_\varphi\left[\left(\frac{2}{r}\frac{df}{dr} + \frac{d^2f}{dr^2}\right)\sin\theta + \frac{1}{r^2\sin\theta}f(r)\big(\underbrace{\cos^2\theta}_{1-\sin^2\theta} - \sin^2\theta - 1\big)\right] = \vec{e}_\varphi\sin\theta\left(\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{2}{r^2}f\right)$$

$$\Delta\vec{A} = \left(\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{2}{r^2}f\right)\sin\theta\,\vec{e}_\varphi = \begin{cases} -\mu_0\rho\omega\sin\theta\,\vec{e}_\varphi\cdot r \;\left(\cdot\theta(R-r)\right) & \text{за } r\leq R \\ 0 & \text{за } r>R \end{cases}$$

$$\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{2}{r^2}f = -\mu_0\rho\omega r$$ Ојлерове диференцијалне јдне

$f = f_h + f_p \qquad f_h \sim r^\alpha \qquad \alpha(\alpha-1) + 2\alpha - 2 = 0 \;;\; \alpha^2 + \alpha - 2 = 0 \;;\; \boxed{\alpha = 1 \text{ или } \alpha = -2}$

$f_p = c r^3 \qquad 6C + 6C - 2C = -\mu_0\rho\omega \Rightarrow C = -\frac{\mu_0\rho\omega}{10}$

$$\vec{A}_< = \left(C_1 r - \frac{\mu_0\rho\omega}{10}r^3\right)\sin\theta\,\vec{e}_\varphi \;;\; \vec{A}_> = C_2\frac{1}{r^2}\sin\theta\,\vec{e}_\varphi$$

Гранични услови:

1) $\vec{A}_< = \vec{A}_> \big|_{r=R}$ 2) $\vec{n}\times(\vec{H}_> - \vec{H}_<)\big|_{r=R} = \vec{i}_{\text{гр.}}$ површинске струје

1) $C_1R - \frac{\mu_0\rho\omega}{10}R^3 = \frac{C_2}{R^2} \Rightarrow \boxed{C_2 = C_1R^3 - \frac{\mu_0\rho\omega}{10}R^5} \Leftarrow f_> = f_<\big|_{r=R}$

2) $\vec{e}_r \times (\text{rot}\,\vec{A}_> - \text{rot}\,\vec{A}_<) = 0$

$$\text{rot}\,\vec{v} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial(\sin\theta\, v_\varphi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right)\vec{e}_r + \left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial(r v_\varphi)}{\partial r}\right)\vec{e}_\theta + \left(\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\vec{e}_\varphi\Big)$$

$$\text{rot}\left(f(r)\sin\theta\,\vec{e}_\varphi\right) = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta f(r)\sin\theta)}{\partial\theta}\vec{e}_r - \frac{1}{r}\frac{\partial(r f\sin\theta)}{\partial r}\vec{e}_\theta$$

$$= \frac{f(r)}{r\cancel{\sin\theta}}2\cancel{\sin\theta}\cos\theta\,\vec{e}_r - \left(\frac{1}{r}f\sin\theta + \frac{df}{dr}\sin\theta\right)\vec{e}_\theta$$

$$= \frac{2\cos\theta}{r}f(r)\vec{e}_r - \left(\frac{df}{dr} + \frac{f}{r}\right)\sin\theta\,\vec{e}_\theta$$

~~2cosθ~~ $\vec{e}_r \times \left(\cancel{(\text{rot}A_>)_r\vec{e}_r} + (\text{rot}A_>)_\theta\vec{e}_\theta - \cancel{(\text{rot}A_<)_r\vec{e}_r} - (\text{rot}A_<)_\theta\vec{e}_\theta\right) = 0$

$\Rightarrow (\text{rot}A_>)_\theta = (\text{rot}A_<)_\theta$

$\frac{df_>}{dr} + \cancel{\frac{f_>}{r}} = \frac{df_<}{dr} + \cancel{\frac{f_<}{r}}\Big|_{r=R}$ јер $f_> = f_<\big|_{r=R}$ из прве гр. услова!

$C_2\left(-\frac{2}{R^3} + \frac{1}{R^3}\right) = C_1 - \frac{3\mu_0\rho\omega}{10}R^2 + C_1 - \frac{\mu_0\rho\omega}{10}R^2 \Rightarrow \boxed{2C_1 - \frac{2\mu_0\rho\omega R^2}{5} = -\frac{C_2}{R^3}}$

$-2C_1R^3 + \frac{2}{5}\mu_0\rho\omega R^5 = -\frac{\mu_0\rho\omega}{10}R^5 + C_1R^3$

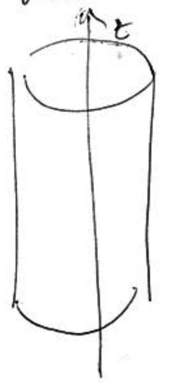
$3C_1R^3 = \frac{\mu_0\rho\omega}{2}R^5 \qquad C_1 = \frac{\mu_0\rho\omega R^2}{6} \qquad C_2 = \frac{\mu_0\rho\omega R^5}{6} - \frac{\mu_0\rho\omega R^5}{10} = \frac{\mu_0\rho\omega R^5}{15}$

$$\boxed{\vec{A} = \begin{cases}\left(\frac{\mu_0\rho\omega R^2}{6}r - \frac{\mu_0\rho\omega}{10}r^3\right)\sin\theta\,\vec{e}_\varphi, & r<R \\ \frac{\mu_0\rho\omega R^5}{15r^2}\sin\theta\,\vec{e}_\varphi, & r\geq R\end{cases}} \qquad \cancel{\vec{B} = \left\{\frac{2\cos\theta}{r}\left(\frac{\mu_0\rho\omega R^2}{6}r - \frac{\mu_0\rho\omega}{10}r^3\right)\sin\theta\,\vec{e}_\varphi\right.}$$

$$\boxed{\vec{B} = \begin{cases} \left(\frac{\mu_0\rho\omega}{3}R^2 - \frac{\mu_0\rho\omega}{5}r^2\right)\cos\theta\,\vec{e_r} - \left(\frac{\mu_0\rho\omega}{3}R^2 - \frac{2\mu_0\rho\omega}{5}r^2\right)\sin\theta\,\vec{e_\theta}, & r\le R \\ \cos\theta\,\frac{2\mu_0\rho\omega}{15}\frac{R^5}{r^3}\vec{e_r} + \frac{\mu_0\rho\omega}{15}\frac{R^5}{r^3}\sin\theta\,\vec{e_\theta}, & r>R \end{cases}}$$

дом. 2; зад бр 5

6. По омотачу бесконачног цилиндра, паралелно његовој оси, тече струја површинске густине $\vec{\iota} = \iota_0 \vec{e_z}$. Наћи векторски потенцијал $\vec{A}$ и густину магнетног поља $\vec{B}$ у свакој тачки простора.



$$\Delta\vec{A} = -\mu_0\vec{J} = 0 \qquad \vec{A} = f(r_c)\,\vec{e_z}$$

$$\Delta\phi = \frac{1}{r_c}\frac{\partial}{\partial r_c}\left(r_c\frac{\partial\phi}{\partial r_c}\right) + \frac{1}{r_c^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\Delta\left(f(r_c)\vec{e_z}\right) = \vec{e_z}\,\Delta f(r_c) = \vec{e_z}\,\frac{1}{r_c}\frac{d}{dr_c}\left(r_c\frac{df(r_c)}{dr_c}\right) = 0$$

$$\Rightarrow r_c\frac{df}{dr_c} = C \;;\; \frac{df}{dr_c} = \frac{C}{r_c} \;/\int \qquad \boxed{f(r_c) = C\ln r_c + D}$$

$$\vec{A}_< = (\cancelto{0}{C_1}\ln r_c + D_1)\,\vec{e_z} \;;\; \vec{A}_> = (C_2\ln r_c + D_2)\,\vec{e_z}$$

гр. услови: 1) $A_< = A_>\big|_{r_c=R}$ $\quad\boxed{D_1 = C_2\ln R + D_2}$

2) $\vec{e_{r_c}}\times(\mathrm{rot}\,\vec{A}_> - \mathrm{rot}\,\vec{A}_<)\big|_{r_c=R} = \iota_0\,\vec{e_z}\cdot\mu_0$ $\qquad \vec{n}\times(\mathrm{rot}\,\vec{A}_> - \mathrm{rot}\,\vec{A}_<) = \vec{\iota}\mu_0$

$$\boxed{\mathrm{rot}\,\vec{v} = \frac{1}{r_c}\left(\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\cdot r_c\right)\vec{e_{r_c}} + \left(\frac{\partial v_{r_c}}{\partial z} - \frac{\partial v_z}{\partial r_c}\right)\vec{e_\varphi} + \frac{1}{r_c}\left(\frac{\partial(r_c v_\varphi)}{\partial r_c} - \frac{\partial v_{r_c}}{\partial\varphi}\right)\vec{e_z}}$$

$$\mathrm{rot}\,\vec{A} = \frac{1}{r_c}\cancelto{0}{\frac{\partial f(r_c)}{\partial\varphi}}\vec{e_{r_c}} - \frac{\partial f(r_c)}{\partial r_c}\vec{e_\varphi} = -\frac{\partial f(r_c)}{\partial r_c}\vec{e_\varphi}$$

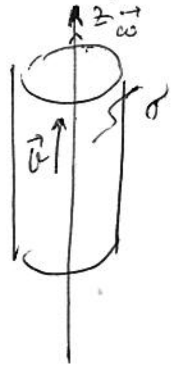
$$\vec{e_{r_c}}\times\left(-\frac{d}{dr_c}(C_2\ln r_c + D_2) + \cancel{\frac{d}{dr_c}}\cancelto{0}{D_1}\right)\vec{e_\varphi}\Big|_R = \mu_0\iota_0\,\vec{e_z}$$

$$-C_2\frac{1}{R} = \mu_0\iota_0 \quad\Rightarrow\quad \boxed{C_2 = -\mu_0\iota_0 R}$$

$$\vec{A} = \begin{cases}\vec{A_0}, & r\le R \\ \vec{A_0} + \mu_0\iota_0 R\ln R\,\vec{e_z} - \mu_0\iota_0 R\ln r_c\,\vec{e_z}, & r>R\end{cases} = \boxed{\begin{cases}\vec{A_0}, & r\le R \\ \vec{A_0} + \mu_0\iota_0 R\ln\frac{R}{r_c}\,\vec{e_z}, & r>R\end{cases}}$$

$$\boxed{\vec{B} = \begin{cases}0, & r\le R \\ \frac{\mu_0\iota_0 R}{r_c}\vec{e_\varphi}, & r>R\end{cases}}$$

7. Бесконачна цилиндрична површ, равномерно наелектрисана површинском густином наелектрисања σ, ротира око своје осе константном угаоном брзином ω и истовремено се креће дуж своје осе брзином v. Наћи векторски потенцијал $\vec{A}$ и магнетно поље $\vec{B}$ у свакој тачки простора.



$$\vec{\omega} = \omega \vec{e}_z \, ; \quad \vec{v} = v\vec{e}_z$$

$$\vec{i} = \sigma \left(v\vec{e}_z + \omega \vec{e}_z \times R\vec{e}_{r_c} \right) = \sigma \left(v\,\vec{e}_z + \omega R\,\vec{e}_\varphi \right)$$

површинске струје

$$\Delta \vec{A} = 0 \qquad \boxed{\vec{A} = A_1(r_c)\vec{e}_\varphi + A_2(r_c)\vec{e}_z}$$

$$\Delta F(r_c,\varphi,z) = \frac{1}{r_c}\frac{\partial}{\partial r_c}\left(r_c \frac{\partial F}{\partial r_c}\right) + \frac{1}{r_c^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{\partial^2 F}{\partial z^2}$$

$$\Delta\vec{A} = \Delta\left(-A_1(r_c)\sin\varphi\,\vec{e}_x + A_1(r_c)\cos\varphi\,\vec{e}_y + A_2(r_c)\,\vec{e}_z \right)$$

$$= -\frac{1}{r_c}\frac{d}{dr_c}\left(r_c A_1'(r_c)\right)\sin\varphi\,\vec{e}_x + A_1(r_c)\sin\varphi\,\frac{1}{r_c^2}\vec{e}_x$$

$$+ \frac{1}{r_c}\frac{d}{dr_c}\left(r_c A_1'(r_c)\right)\cos\varphi\,\vec{e}_y - A_1(r_c)\cos\varphi\,\frac{1}{r_c^2}\,\vec{e}_y$$

$$+ \frac{1}{r_c}\frac{d}{dr_c}\left(r_c A_2'(r_c)\right)\vec{e}_z$$

$$= \frac{1}{r_c}\left(A_1' + r_c A_1''\right)\vec{e}_\varphi - A_1\frac{1}{r_c^2}\vec{e}_\varphi + \overbrace{\frac{1}{r_c}\left(A_2' + A_2'' r_c\right)\vec{e}_z}^{\frac{1}{r_c}\frac{d}{dr_c}\left(r_c A_2'(r_c)\right)\vec{e}_z}$$

$$\frac{d}{dr_c}\left(r_c A_2'(r_c)\right) = 0$$

$$A_2'(r_c) = \frac{C_2}{r_c}$$

$$\boxed{A_2(r_c) = C_2 \ln r_c + D_2}$$

$$\Rightarrow \quad A_1'' + \frac{1}{r_c}A_1' - \frac{1}{r_c^2}A_1 = 0 \quad \text{и} \quad A_2'' + \frac{1}{r_c}A_2' = 0$$

$$A_1 \sim r_c^{\alpha} \qquad \alpha(\alpha-1) + \alpha - 1 = 0$$

$$\alpha^2 - 1 = 0 \Rightarrow \alpha = \pm 1$$

$$A_1 = C_1 r_c + \frac{D_1}{r_c}$$

$$\frac{dA_2'}{A_2'} = -\frac{dr_c}{r_c} \quad \Big/ \int$$

$$\ln A_2' = -\ln r_c + \text{const.}$$

$$A_2' = \frac{1}{r_c}\cdot\text{const.}$$

$$\boxed{A_2 = C_2 \ln r_c + D_2}$$

$$\vec{A} = \begin{cases} \overbrace{C_1 r_c}^{A_1}\,\vec{e}_\varphi + \overbrace{C_2}^{A_2}\,\vec{e}_z \, , & r \le R \\ \underbrace{\frac{C_3}{r_c}}_{A_1}\,\vec{e}_\varphi + \underbrace{\left(C_4 \ln r_c + C_5\right)}_{A_2}\vec{e}_z \, , & r > R \end{cases}$$

$$\boxed{\begin{aligned} &\operatorname{rot}\left(A_1(r_c)\vec{e}_\varphi + A_2(r_c)\vec{e}_z\right) \\ &= -\frac{dA_2}{dr_c}\vec{e}_\varphi + \frac{1}{r_c}\frac{d}{dr_c}\left(r_c A_1(r_c)\right)\vec{e}_z \end{aligned}}$$

Гр. услови:

$1^\circ \quad \vec{A}_< = \vec{A}_> \big|_{r_c = R}$ $\qquad 2^\circ \quad \vec{n} \times \left(\vec{H}_> - \vec{H}_<\right)\big|_{r=R} = \vec{i}$

$$C_1 R\,\vec{e}_\varphi + C_2\,\vec{e}_z = \frac{C_3}{R}\vec{e}_\varphi + \left(C_4 \ln R + C_5\right)\vec{e}_z$$

$$\Rightarrow \quad \boxed{C_1 R = \frac{C_3}{R}} \qquad \boxed{C_2 = C_4 \ln R + C_5}$$

$2^\circ \quad \vec{e}_{r_c} \times \left(\operatorname{rot}\vec{A}_> - \operatorname{rot}\vec{A}_<\right)\big|_{r=R} = \mu_0 \sigma \left(v\vec{e}_z + \omega R\vec{e}_\varphi\right)$

$$\vec{e}_{r_c} \times \left(-\frac{d}{dr_c}\left(C_4 \ln r_c + \cancel{C_5}\right)\vec{e}_\varphi + \frac{1}{r_c}\cancel{\frac{d}{dr_c}\left(r_c \cdot \frac{C_3}{r_c}\right)}\vec{e}_z + \cancel{\frac{d}{dr_c}\left(C_2\right)}\vec{e}_\varphi - \frac{1}{r_c}\frac{d}{dr_c}\left(r_c C_1 r_c\right)\vec{e}_z \right)\Bigg|_{r=R} = \mu_0\sigma\left(v\vec{e}_z + \omega R\vec{e}_\varphi\right)$$

$$-C_4 \frac{1}{R}\,\vec{e}_z + 2C_1\,\vec{e}_\varphi = \mu_0 \sigma v\,\vec{e}_z + \mu_0 \sigma \omega R\,\vec{e}_\varphi$$

$\boxed{C_4 = -\mu_0 R \sigma v}$ $\boxed{C_1 = \frac{1}{2}\mu_0 \sigma \omega R}$

$C_3 = C_1 R^2 = \boxed{\frac{1}{2}\mu_0 \sigma \omega R^3}$

$$\boxed{\vec{A} = \begin{cases} \frac{1}{2}\mu_0 \sigma \omega R r\, \vec{e}_\varphi - \mu_0 R \sigma v \ln R\, \vec{e}_z, & r \le R \\ \frac{1}{2}\mu_0 \sigma \omega R^3 \frac{1}{r}\, \vec{e}_\varphi - \mu_0 R \sigma v \ln r\, \vec{e}_z, & r > R \end{cases}}$$

ii $C_2 = 0$ $C_5 = +\mu_0 R \sigma v \ln R$

$$\vec{A} = \begin{cases} \frac{1}{2}\mu_0 \sigma \omega R r\, \vec{e}_\varphi, & r \le R \\ \frac{1}{2}\mu_0 \sigma \omega R^3 \frac{1}{r}\, \vec{e}_\varphi + \mu_0 R \sigma v \ln\left(\frac{R}{r}\right) \vec{e}_z, & r > R \end{cases}$$

$$\boxed{\vec{B} = \begin{cases} \mu_0 \sigma \omega R\, \vec{e}_z, & r \le R \\ \mu_0 R \sigma v \frac{1}{r}\, \vec{e}_\varphi, & r > R \end{cases}}$$